Backscattering: an overlooked effect of General Relativity?

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The total flux of outgoing radiation in a strong gravitational field decreases due to backscattering if the sources are close to an apparent horizon. It can cause detectable changes in the shape of signals. Backscattering could well be of relevance to astrophysics and would constitute a new test of the validity of general relativity. An explicit bound for this effect is derived for scalar fields.

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Backscattering can be described as a phenomenon which causes the radiation of a massless field to disperse outside those null cones defined by the initial impulse. It is known in older mathematical literature as the 'breakdown of Huyghens principle' [1] and it is interesting for a number of reasons. First, as a non-Minkowski spacetime effect, it offers a new way of testing general relativity. Second, it can be important in astrophysics, both in order to explain the radiation coming to us from regions adjoining a black hole and to infer information about the sources of the radiation. Third, a strict upper bound on the magnitude of the backscattering effect, such as we derive here, offers numerical relativists an independent test of the correctness of their numerical codes.

Our model is a spherically symmetric massless scalar field emitted from a region near to, but outside, a non-rotating black hole. These simplifying assumptions should not seriously restrict the validity of our conclusions. For example, the propagation of electromagnetic fields, we believe, should obey inequalities similar to the ones derived below. These estimates, while they break down close to the horizon, allow us to distinguish the region in which the backscattering may play a significant role from that in which it is of no importance. They offer, to our knowledge, the first quantitative measure of this strong field effect. Others who work on backscattering adopt quite different approaches [2].

We consider a foliation of the spacetime by using the polar gauge slicing condition, $trK = K_r^r$; that is, with a diagonal line element

$$ds^2 = -\beta(R,t)\gamma(R,t)dt^2 + \frac{\beta(R,t)}{\gamma(R,t)}dR^2 + R^2(r,t)d\Omega^2 , \tag{1}$$

where t is the time, R is a radial coordinate which coincides with the areal radius and $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the line element on the unit sphere, $0 \leq \phi < 2\pi$ and $0 \leq \theta \leq \pi$; β and γ go to $+1$ at infinity.

For a massless scalar field the stress-energy is $T_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - g_{\mu\nu}\nabla_\alpha\phi\nabla^\alpha\phi/2$. The matter energy density is $\rho = -T_0^0$ and the matter current density is $J = -T_{0R}/\beta$. $(\partial_0 \pm \gamma\partial_R)$ are the outgoing and ingoing null directions. We define radiation amplitudes

$$h_+(R,t) = h(R,t) = \frac{1}{2\gamma}(-\partial_0 + \gamma\partial_R)(R\phi) \tag{2}$$

$$h_-(R,t) = h(-R,t) = \frac{1}{2\gamma}(\partial_0 + \gamma\partial_R)(R\phi). \tag{3}$$

One can show [5] that

$$\beta(R) = e^{-8\pi \int_R^\infty \frac{dr}{r}\left[(h_+ - \hat{h})^2 + (h_- - \hat{h})^2\right]} , \tag{4}$$

and

$$\gamma(R) = 1 - \frac{2m_0}{R} + \frac{1}{R}\int_R^\infty [1 - \beta(r)]dr , \tag{5}$$

where $\hat{h} = -\frac{1}{2R}\int_R^\infty dr[h_+(r) + h_-(r)] = \frac{1}{2}\phi$. γ can be expressed in the following useful form [5]

$$\gamma(R) = \left(1 - \frac{2m_0}{R} + \frac{2m_{ext}(R)}{R}\right)\beta(R) \tag{6}$$

where m_0 is the asymptotic (total) mass and m_{ext} is a contribution to the asymptotic mass coming from the exterior of a sphere of a radius R,

$$m_{ext}(R) = 4\pi \int_R^\infty \frac{\gamma}{\beta}\left([h_+(r) - \hat{h}]^2 + [h_-(r) - \hat{h}]^2\right) dr . \tag{7}$$

The scalar field equation $\nabla_\mu\partial^\mu\phi = 0$ can be written as a single first order equation on a 'symmetrized' domain $-\infty \leq R \leq \infty$ [5] by writing $h(R) = h_+(R)$ and $h(-R) = h_-(R)$ as

$$(\partial_0 + \gamma\partial_R)h = (h - \hat{h})\frac{\gamma - \beta}{R} . \tag{8}$$

Eq. (8), together with the definitions of $h, \hat{h}, \beta$, and γ, is equivalent to the Einstein equations coupled to the scalar

field. The external mass changes along an outgoing null cone according to

$$(\partial_0 + \gamma\partial_R)m_{ext}(R) = -8\pi\gamma^2(h_- - \hat{h})^2 \ . \qquad (9)$$

The polar gauge allows us to express the metric directly in terms of the matter, Eqs. (4 - 5), where all the integrals are in the exterior region. The local and global Cauchy problems for the above system are solvable in an external region bounded from the interior by a null cone ([4], [5]).

Following Eq.(6), $(\gamma - \beta)/R = -2m(R)R\beta/|R|^3$ where $m(R) = m_0 - m_{ext}(R)$ is the Hawking mass at a radius R. Eq.(8) gives a 'red-shift' due to the h term on the right-hand-side (determined by the mass function, $m(R)$, rather than the Schwarzschild mass, m_0) and a 'backscattering' due to the $\hat{h}$ term.

Let us define

$$\ln\left[1 - \frac{2\tilde{m}(R)}{R}\right] = -\int_R^\infty \frac{2m(r)dr}{r^2(1 - 2m(r)/r)} \ , \qquad (10)$$

where the integral is taken along an outgoing null ray. This allows us to rewrite Eq.(8) as

$$(\partial_0 + \gamma\partial_R)(1 - \frac{2\tilde{m}(R)}{R})h = \hat{h}(1 - \frac{2\tilde{m}(R)}{R})\frac{2m(R)R\beta}{|R|^3} \ . \qquad (11)$$

Thus $(1 - 2\tilde{m}(R)/R)$ is the redshift factor and the right-hand-side of Eq.(11) determines the backscattering.

It is natural to write Eq.(4) as

$$\beta(R) = e^{-8\pi\left(\int_R^\infty + \int_{-R}^{-\infty}\right)\frac{\beta}{\gamma r}\frac{\gamma}{\beta}(h-\hat{h})^2 dr} \ , \qquad (12)$$

and by using Eq.(6) we get that

$$\frac{\beta}{\gamma r} = \frac{1}{r - 2m(r)} \le \frac{1}{r - 2m_0} \ . \qquad (13)$$

The factor $\beta/\gamma r$ can be taken out of the integral in Eq.(12) and replaced with its value at $r = R$. The remainder is then essentially right hand side of Eq.(7). Let us choose an ϵ and an R_A such that $m_{ext}(R_A)/m_0 < \epsilon$ and $R_A > 2m_0(1 + \epsilon)$. For any $R \ge R_A$ we obtain

$$1 \ge \beta(R) \ge \beta(R_A) \ge e^{-2\frac{m_{ext}(R_A)}{R_A - 2m_0}} \simeq 1 - O(m_{ext}/\epsilon m_0) \ . \qquad (14)$$

In the same vein, using Eq.(6) one gets $\gamma(R) \simeq 1 - 2m_0/R + O(m_{ext}/\epsilon m_0)$. Thus the effect of the matter on the geometry can be controlled. Notice that in regions sufficiently close to the horizon, even if $m_{ext}/m_0 \ll 1$, a small cloud of matter can still strongly influence the geometry. In what follows, however, we will restrict our attention to the region outside $R = 3m_0$, since only in that region of spacetime can we get sensible analytic estimates. This is equivalent to choosing $\epsilon \ge 1/2$.

For $R > 3m_0$ the scalar wave equation Eq.(11) can be approximated as

$$(\partial_0 + \gamma\partial_R)(1 - \frac{2m_0}{R})h = \hat{h}(1 - \frac{2m_0}{R})\frac{2m_0 R}{|R|^3} \ , \qquad (15)$$

with the error terms of order m_{ext}/m_0. In the limit of $m_{ext}/m_0 \ll 1$ the equation describes a scalar field propagating on a fixed Schwarzschild background.

One can show, improving a coefficient in an inequality of [5], that

$$|\hat{h}| \le \frac{\sqrt{m_{ext}}}{R^{1/2}\sqrt{8\pi(1 - \frac{2m_0}{R})}}. \qquad (16)$$

(A similar estimate with m_0 rather than m_{ext}, appears in [6].) Let us introduce the Regge-Wheeler coordinate [3] $r^* = R + 2m_0\ln(R/2m_0 - 1)$ so that $\gamma\partial_R = \partial_{r^*}$. The solution of Eq.(15) can be estimated above and below by solutions of

$$(\partial_0 + \partial_{r^*})\left((1 - \frac{2m_0}{R})h\right) = {}^+_- \frac{2m_0(1 - \frac{2m_0}{R})\sqrt{m_{ext}}}{R^{5/2}\sqrt{8\pi(1 - \frac{2m_0}{R})}}. \qquad (17)$$

Eqns.(17) are solved by

$$\left(1 - \frac{2m_0}{R(r)}\right)h(r^*,t) = h_0(r^* - t)$$
$$\pm \sqrt{\frac{m_{ext}}{2\pi}}m_0\int_{(r-t,0)}^{(r,t)}\frac{\left(1 - \frac{2m_0}{R}\right)}{R^2\sqrt{R - 2m_0}}dv \ , \qquad (18)$$

here the dv in Eq.(18) is dr^*, with $dR/dr^* = 1 - 2m_0/R$. If we ignore the second term then the standard redshift is obtained,

$$h(r^*(0) + \tau, \tau) = \frac{\left(1 - \frac{2m_0}{R(r^*(0))}\right)}{\left(1 - \frac{2m_0}{R(r^*(0)+\tau)}\right)}h(r^*(0),0) \ , \qquad (19)$$

while the integral in Eq.(18) can be solved to give

$$\sqrt{\frac{m_{ext}}{2\pi}}m_0\int\frac{dR}{R^2\sqrt{(R - 2m_0)}} =$$
$$\sqrt{\frac{m_{ext}}{16\pi m_0}}\left[\sqrt{\frac{2m_0}{R}\left(1 - \frac{2m_0}{R}\right)} + \arctan\sqrt{\frac{R - 2m_0}{2m_0}}\right] \ . \qquad (20)$$

Thus the total backscattering is bounded by a term of order $\sqrt{m_{ext}/m_0}$. In the limit where m_{ext}/m_0 is small, we recover the usual gravitational redshift along an outgoing null ray with an error of order $\sqrt{m_{ext}/m_0}$. This holds even in a strong gravitational field and no 'quasi-static' assumption need be made. We need not assume $R \gg m_0$, but only $R \ge 3m_0$.

Let us assume that initial data at $t = 0$ represents a pure outgoing wave. In other words, $h_-(t = 0) \equiv 0$. We select an $R_A \ge 3m_0$ on the initial slice such that $m_{ext}(R_A)/m_0 \ll 1$. Finally we assume, in addition to $\phi \to 0$ at infinity, that $\phi = 0$ for $R < R_A$. This guarantees that the radiation is bounded away from the black

hole. Consider the future outgoing lightray from $(R_A, 0)$. This is well outside any event horizon, which would be at approximately $R = 2m_0$, so this lightray is really outgoing all the way to null infinity. Note that $m_{ext}(R_A, 0)$ is the maximum value of m_{ext} over the whole wedge bounded by the initial slice and the outgoing null ray. We will estimate h_- and $\hat{h}$ along this null ray. The integration of Eq.(9) along this ray yields an estimate of the total energy flux across this surface in the inward direction. This will be the total energy loss from the outgoing wave due to backscatter.

Choose a point on the null ray from R_A and label it by (R_1, T_1). To calculate h_- at this point, consider the ingoing future null ray which passes through this point and integrate Eq.(17) along this ingoing ray. This will start from the initial hypersurface at some point $(R_2, 0)$ with $R_2 > R_A$. Along this null ray R monotonically decreases while m_{ext} monotonically increases.

To get an explicit estimate, the integral in Eq.(18) can be further approximated; since R monotonically decreases along the ingoing lightray, we can replace the $\sqrt{1 - 2m_0/R}$ by $\sqrt{1 - 2m_0/R_1}$. This yields

$$\int_{R_2}^{R_1} \frac{dR}{R^2\sqrt{R - 2m_0}} \leq$$
$$\sqrt{\frac{R_1}{R_1 - 2m_0}} \int \frac{dR}{R^{5/2}} \leq \frac{2}{3R_1^{3/2}} \sqrt{\frac{R_1}{(R_1 - 2m_0)}} . \quad (21)$$

Thus we arrive at

$$|\gamma h_-(R)| \leq \frac{2}{3} \sqrt{\frac{m_{ext}}{2\pi m_0}} \left[\frac{m_0^{3/2}}{R\sqrt{R - 2m_0}} \right] . \quad (22)$$

We can write, using Eq.(3) and (22),

$$(\partial_0 + \partial_{r*})(R\hat{h}) = \gamma h_- \leq \frac{2}{3} \sqrt{\frac{m_{ext}}{2\pi m_0}} \left[\frac{m_0^{3/2}}{R(R - 2m_0)^{1/2}} \right] \quad (23)$$

and a similar inequality with a minus sign to give a lower bound of $\hat{h}$.

These equations are integrated along the outgoing null ray from R_A to give

$$|R\hat{h}(R, t)| \leq |R\hat{h}(R_A, 0)| + \frac{2}{3} \sqrt{\frac{m_{ext}}{2\pi}} m_0 \int \frac{dR}{(R - 2m_0)^{3/2}} , \quad (24)$$

Since we demand that $\phi(R_A, 0) = 0$, the first term in Eq.(24) vanishes. Thus we can bound $\hat{h}$ by

$$|\hat{h}(R, t)| \leq \frac{4}{3} \sqrt{\frac{m_{ext}}{2\pi m_0}} \frac{m_0^{3/2}}{R(R_A - 2m_0)^{1/2}} -$$
$$\frac{4}{3} \sqrt{\frac{m_{ext}}{2\pi m_0}} \frac{m_0^{3/2}}{R(R - 2m_0)^{1/2}} . \quad (25)$$

The last term in Eq.(25) is strictly larger than $|h_-|$ as given by Eq.(22) if $R > R_A > 4m_0$ (i. e., when *all* radiation is placed outside $4m_0$). Thus

$$|\hat{h} - h_-| \leq |\hat{h}| + |h_-| \leq \frac{4}{3} \sqrt{\frac{m_{ext}}{2\pi m_0}} \frac{m_0^{3/2} R_A}{R(R_A - 2m_0)^{3/2}} . \quad (26)$$

Therefore, from Eq.(9) the total change in m_{ext} satisfies

$$\Delta m_{ext} \leq m_{ext} \frac{16}{9} \left(\frac{2m_0}{R_A} \right)^2 \left(\frac{1 - m_0/R_A}{1 - 2m_0/R_A} \right) . \quad (27)$$

From this expression we can see how sensitive the amount of backscattering is to the location of the innermost null cone. This estimate becomes meaningless if $R_2 \approx 3.5m_0$ because $\Delta m_{ext} > m_{ext}$. However if $R_A = 6m_0$ we get that less than 25% of the total energy in the exterior field is backscattered. In the case of a neutron star, where $2m_0/R \leq 0.1$ (on the surface of the star), we have an upper bound for the backscattered energy of 2% of m_{ext}.

In deriving the bound Eq.(27), a number of truncations and approximations were used. While most of them are sharp in the sense that configurations exist that turn the inequality into an equality, we do not believe that all of them can be sharp simultaneously. Therefore Eq.(27) is clearly an overestimate. We present it here, not because it is the best that can be done with this technique, but because it is simple to derive, it is in analytic form, and is easy to interpret physically.

For example, in Eq.(21) we ignored the term depending on R_2. We can include this term if the initially outgoing pulses are far enough from the apparent horizon [7], so as to sharpen the estimate to m_{ext}

$$\Delta m_{ext} \leq \frac{16\alpha^2}{9} \left(\frac{2m_0}{R_2} \right)^2 \frac{(1 - m_0/R_2)}{1 - 2m_0/R_2} m_{ext} , \quad (28)$$

where $\alpha^2 \simeq 0.4$. Applying this to the case of neutron stars, when the conditions assumed above hold true, we find that the maximal amount of backscattered radiation cannot exceed 1 percent.

Eq.(18) can be integrated in closed form, (Eq.(20)), but in Eq.(21) we approximated it to get a simpler expression because we needed to integrate the result again. Numerical integration would obviate the need for this approximation entirely, but the result would be much less transparent.

The tail term can also be bounded using this method. The initially outgoing field, h_+, generates a weaker ingoing field, h_-, which enters the 'no-radiation' zone behind the wavefront. This, in turn, scatters again off the gravitational field to generate a new outgoing field, which turns up at null infinity at a later time, after the first burst of outgoing radiation has gone. This is the so-called 'tail' term. Eq.(25) gives us an estimate for $\hat{h}$ in the 'no-radiation' zone. Substituting this back into the scalar wave equation (15), an estimate of the second order h_+ along the outgoing null ray is

$$|\gamma h_+(R,t)| \leq$$

$$\tfrac{4}{3} \sqrt{\frac{m_{ext}}{2\pi m_0}} \frac{m_0^{5/2}}{\sqrt{R_A - 2m_0}} \int_{R_A}^{R} \left[\frac{1}{R^3} - \frac{\sqrt{R_A - 2m_0}}{R^3 \sqrt{R - 2m_0}} \right] dR . \quad (29)$$

In the limit as $R \to \infty$ along the null cone we get

$$|h_+(\infty, \infty)| \leq \frac{4}{9} \sqrt{\frac{m_{ext}}{2\pi m_0}} \frac{(m_0/R_2)^{5/2}}{\sqrt{1 - 2m_0/R_2}} . \quad (30)$$

This should be compared with the leading term in (18), h_0. Using the definition of the external mass m_{ext}, we get that the tail term is smaller than the leading term by a factor $(m_0/R_0)^2$.

We expect that photons will behave in a way similar to the massless spin zero field that we have discussed here. Gravitational physics predicts two phenomena for a radiating plasma surrounding a black hole or a neutron star: Redshift diminishes the intensity and frequency of the outgoing radiation while the total energy in the radiation (as measured by the mass function) remains unchanged. Backscattering, the other effect, weakens the overall outgoing radiation, so that the total energy that reaches infinity is reduced. In addition, backscattering changes the shape of extended signals - the leading part weakens while the rest of the impulse gains in intensity. This conclusion follows from a careful analysis of (8) and is supported by the numerical results of [7], which show that up to 10% of the radiation emitted at $R = 3m_0$ is 'shifted' inside the main impulse; even if it reaches infinity, it does so with a significant delay.

There are two astrophysical situations where those two different aspects of backscattering will play a role. The net efficiency of the black hole - matter system is reduced below what is expected when only the redshift factors are taken into account. This may be of significance in modelling 'too faint' galactic nuclei fuelled by black holes, such as the nuclei of M87 [8]. Second, backscattering may leave imprints in X ray bursts [9] resulting from energetic processes on the surface of a neutron star or in collisions of compact bodies, such as neutron star - neutron star, neutron star - black hole, or black hole - planetoid mergers. Backscattering will deform the peak and contribute a tail term to the radiation emitted from such short-lived sources. Gamma ray bursts (GRB) are believed by some [10] to arise from such collisions and should reveal traces of backscattering. The absence of these effects would give great support to the fireball scenario of GRB's [11].

Another more immediate application is in numerical relativity. Much work has been done in constructing codes to analyse the Einstein - massless scalar field model [12]. Bounds on the magnitude of the backscattering effect, such as derived here, would offer numerical relativists a reliable test of the correctness and long-time stability of their codes.

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